DIAS Delhi Institute for Administrative Services
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ALL INDIA TEST SERIES CSE-2023

Candidate 's Information

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4. SUBJECT:- Mechanics
5. DATE:- 28-07-2024.

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Q.NO	MARKS
1.	23 1/2
2.	18
3.	29
4.	28
5.	33 1/2
6.	
7.	
8.	

TOTAL MARKS	132

250

EXAMINER SIGNATURE

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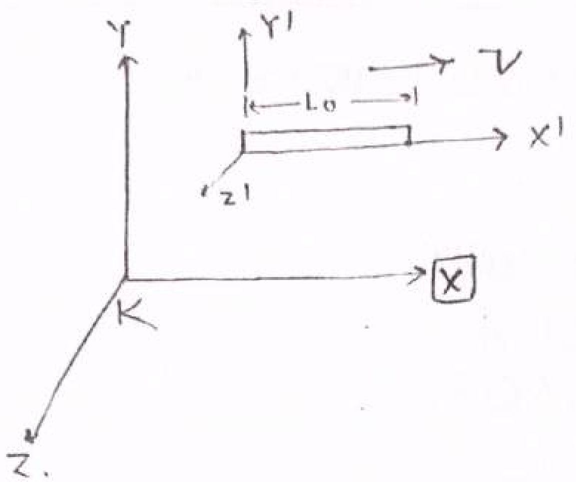
Question - 1

Ans 1(a)



The proper length in Rest frame of Rod be Lo which gets contracted in K' frame moving with respect to the K-frame with velocity V.

New Length $L' = L_0 - \frac{0.5 L_0}{100} = 0.995 L_0$

∵ Length contracted $L' = \frac{L_0}{\gamma}$

with $\gamma = \left(1 - \frac{V^2}{C^2}\right)^{-\frac{1}{2}} \Rightarrow \gamma = \frac{L_0}{L'} = \frac{L_0}{0.995 L_0} = 1.005$

$$\left(1 - \frac{V^2}{C^2}\right)^{-\frac{1}{2}} = 1.005$$

$$1 - \frac{V^2}{C^2} = 0.990$$

$$\frac{V^2}{C^2} = 0.01$$

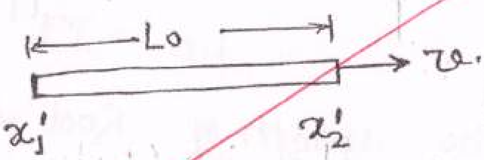
$$\boxed{V = 0.1 C}$$

So, at $v = 3\times10^7$ m-sec^{-1}, the length L_0 will get contracted to 0.5% Less than L_0.

Length contraction and time dilation are proved through Muon decay.

$6\frac{1}{2}$ / 10

Ans 1(b) Length of rod = Length separation by two time markers.



L = product of velocity × time interval between passing of two end points at same marker

Let a rod of proper length L_0 passes through a point 'x'.

The time interval $\Delta t' = t_2' - t_1'$ for same x.

$$= \gamma\left(t_2 - \frac{Vx}{c^2}\right) - \gamma\left(t_1 - \frac{Vx}{c^2}\right)$$

$$\Delta t' = \gamma(t_2 - t_1) = \gamma\Delta t$$

प्रश्न संख्या (Question No.)

DIAS

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Here we've used time dilation formulae between two frames.

Length of rod $L_0 = v\Delta t'$ ⟶ (1)

Now for a stationary observer the time period will be different.

It will be $\gamma\left(\frac{\Delta t'}{\gamma}\right)$ and hence the

length $L' = \gamma\left(v\frac{\Delta t'}{\gamma}\right)$ ⟶ (2)

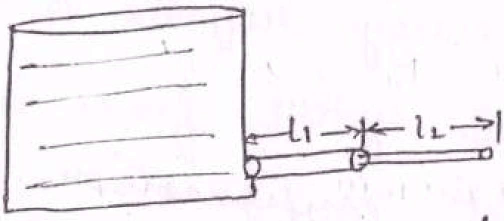
eqn $\frac{(1)}{(2)}$: $\frac{L_0}{L'} = \gamma$

$L_0 = \gamma L'$ where $\gamma > 1$, $L' < L_0$

or the length of Rod in stationary frame appears contracted when rod is moving w.r.t. velocity $\vec{v}$.

6/10

Ans 1(c)



Given: Length of first capillary $l_1 = 20$ cm

" " second " $l_2 = 10$ cm

Diameter of first capillary $d_1 = 2$ mm

Radius $r_1 = 1$ mm

$r_2 = 1$ mm.

Flow out of tank $= Q = 0{\cdot}2$ cc of water.

In series $Q_1 = Q_2$.

Or $\because Q = \frac{\pi P r^4}{8\eta l}$ (Poiseulli's formula)

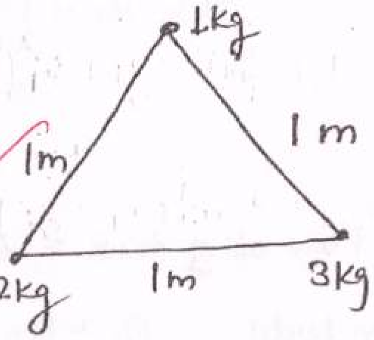
$$\frac{\pi P_1 r_1^4}{8\eta l_1} = \frac{\pi P_2 r_2^4}{8\eta l_2}$$

Hence the flow will remain same as the pipes are connected in series,

Poiseulli's formula is limited to NARROW tube only.

Ans 1(d) Gravitational potential energy of a system is defined as energy required to bring each particle of system from infinity to the present configuration.

$$U = -\int_{\infty}^{r} \vec{F}\cdot d\vec{r} = -\int_{-\infty}^{r} \frac{GMm}{r^2}\,dr = -\frac{GMm}{r}$$



$$U = -G\left[\frac{m_1m_2}{r_{12}} + \frac{m_2m_3}{r_{23}} + \frac{m_3m_1}{r_{31}}\right]$$

$$= -\frac{G}{r}\left[2+6+3\right]$$

$$= -\frac{6.67\times10^{-11}\times 11}{1}$$

$$\boxed{U = -7.337\times10^{-10}\ \text{Joules}}$$

This energy is due to the configuration of system.

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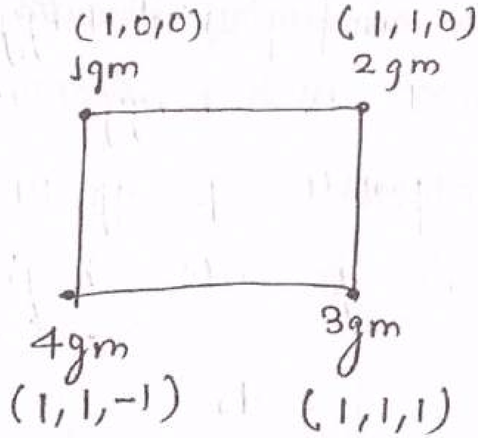
Ans 1(e) **Inertia tensor:**



The components of the tensor –

$I_{xx} = \sum_{i=1}^{4} m_i (y_i^2 + z_i^2)$

$= 1(0) + 2(1) + 3(2) + 4(2) = 16 \cdot gm\text{-}cm^2$

$I_{yy} = \sum_{i=1}^{4} m_i (z_i^2 + x_i^2) = 1(1) + 2(1) + 3(2) + 4(2)$

$= 17 \cdot gm\text{-}cm^2$

$I_{zz} = \sum_{i=1}^{4} m_i (x_i^2 + y_i^2) = 1 + 2(2) + 3(2) + 4(2)$

$= 19 \; gm\text{-}cm^2$

$I_{xy} = I_{yx} = \sum_{i=1}^{4} m_i x_i y_i = 1(0) + 2(1) + 3(1) + 4(1)$

$= 9 \; gm\text{-}cm^2$

$I_{yz} = I_{zy} = \sum_{i=1}^{4} m_i y_i z_i = 1(0) + 2(0) + 3(1) + 4(-1)$

$= 3 - 4$

$= -1 \cdot gm\text{-}cm^2$

$I_{zx} = I_{xz} = \sum_{i=1}^{4} m_i x_i z_i = 1(0) + 2(0) + 3(1) + 4(-1)$

$= -1 \; gm\text{-}cm^2$

$$\text{so } I = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} = \begin{pmatrix} 16 & -9 & +1 \\ -9 & 17 & -1 \\ 1 & -1 & 19 \end{pmatrix}.$$

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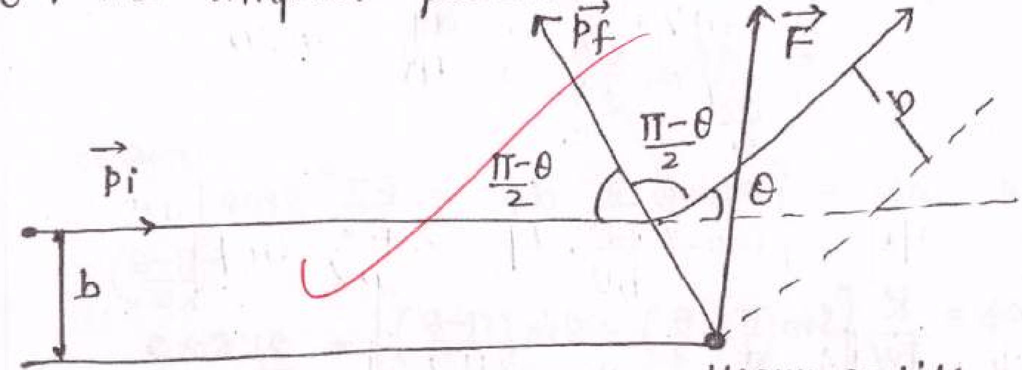
The moment of inertia tensor is symmetric tensor with 6 independent components out of total 9.

Question - 2

Ans 2(a) considering elastic scattering of a particle by a heavy particle. Let the mass of incident particle be 'm' which scatters at an angle 'θ'. The impact parameter be 'b'.

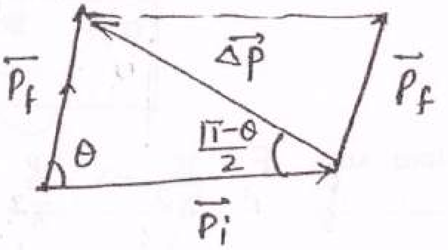


Rutherford Scattering

Let the initial momentum of particle be $\vec{P_i}$ & post scattering it is $\vec{P_f}$. The scattering is elastic and hence:

$$|\vec{P_i}| = |\vec{P_f}|$$

Change in momentum $= \Delta\vec{p}$



Using sine-law:

$$\frac{\Delta P}{\sin\theta} = \frac{|\vec{P_f}|}{\sin\left(\frac{\pi-\theta}{2}\right)} \quad \text{or} \quad \Delta p = \frac{mv}{\cos\frac{\theta}{2}} \cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\boxed{\Delta p = 2mv\sin\frac{\theta}{2}} \longrightarrow (1)$$

This change of momentum is due to the IMPULSE felt by particle.

$$= \int \vec{F} \cdot dt = \int F\cos\phi \cdot dt$$

changing to ϕ coordinates: $\int F\cos\phi \left(\frac{dt}{d\phi}\right) d\phi$ → (2)

$\because$ Angular momentum is constant:

$$J = mvr = mr^2 \frac{d\phi}{dt} = mvb$$

$$\Rightarrow \frac{d\phi}{dt} = \frac{vb}{r^2}$$

So $\Delta p = \int F\cos\phi \frac{r^2}{bv} \cdot d\phi = \frac{Fr^2}{bv} \sin\phi \Big|_{-\left(\frac{\pi-\theta}{2}\right)}^{(\pi+\theta)/2}$.

$$\Delta p = \frac{K}{bV}\left[\sin\left(\frac{\pi}{2} + \frac{\theta}{2}\right) + \sin\left(\frac{\pi-\theta}{2}\right)\right] = \frac{2K\cos\frac{\theta}{2}}{bV}$$

or $2mv\sin\frac{\theta}{2} = \frac{2K}{bV}\cos\frac{\theta}{2}$ where $F = \frac{K}{r^2}$

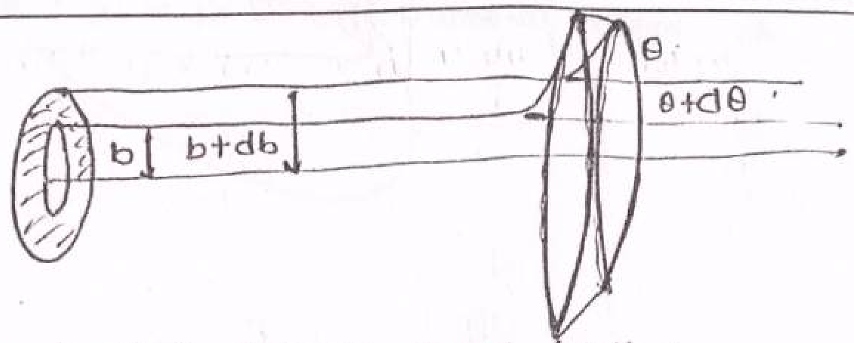
or $b = \frac{K\cot\theta/2}{mv^2}$

$$\boxed{b = \frac{K}{2E_\alpha}\cot\frac{\theta}{2}}$$

Now if $F = \frac{K}{r^2} \Rightarrow \frac{K}{r^2} = \frac{2Ze^2}{4\pi\epsilon_0 r^2}$ or $\boxed{K = \frac{2Ze^2}{4\pi\epsilon_0}}$

which is the desired relation,

Now,



The flux incident between b and b+db :

$dN = (2\pi b\,db) I$ —— (i)

The particles scattered between θ and θ+dθ

$dN = I\,\sigma(\Omega)\,d\Omega = 2\pi \sin\theta\, I\, \sigma(\Omega)\, d\theta$ —— (ii)

(i) must be equal to (ii)

so $2\pi b\,db = \sigma(\Omega)\, 2\pi \sin\theta\, d\theta$

or $db\,b = \sigma(\Omega) \sin\theta\, d\theta$

$$\boxed{\sigma(\Omega) = \frac{-b}{\sin\theta}\left|\frac{db}{d\theta}\right|}$$

−ve sign is for as θ↑ b↓.

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NUMERICAL : Z = 82, A = 207, E_α = 7.7 MeV.
θ = 30°

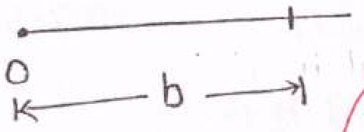
so $b = \frac{K}{2E_\alpha}\cot 30° = \frac{2Ze^2}{4\pi\epsilon_0}\cdot\frac{1}{2E_\alpha}\cdot\sqrt{3}$.

$$= \frac{2\times 82\times(1.6\times10^{-19})^2\times\sqrt{3}}{2\times 7.7\times(1.6\times10^{-13})}\times 9\times10^{9}$$

$b = 2.66\times10^{-12}$

so Impact parameter $\boxed{b = 2.66 \times 10^{-12}\, m}$

Ans 2 (b) Given central force $\vec{F} = \frac{-K}{r^3}\hat{r}$.



$$m\frac{dv}{dt} = \frac{-K}{r^3}$$

$$\frac{dv}{dt} = \frac{-K}{mr^3} \quad \text{or} \quad \frac{d^2r}{dt^2} = \frac{-K}{mr^3}.$$

In straight line: $\frac{d^2x}{dt^2} = \frac{-K}{mx^3}$.

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{-K}{mx^3}$$

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Ans 2(c) Escape velocity is defined as velocity of a particle attaining which it escapes the gravitational field of its celestial body.

for this: Gravitational potential energy $U = -\frac{GMm}{R}$

and $KE = \frac{1}{2}mv_e^2$

so for escaping $\frac{1}{2}mv_e^2 = +\frac{GMm}{R}$

$$v_e = \sqrt{\frac{2GM}{R}}$$

For moon: $M = 7.4 \times 10^{22}$ kg

$R = 1.74 \times 10^6$ m

$$v_e = \left(\frac{2 \times 6.67 \times 10^{-11} \times 7.4 \times 10^{22}}{1.74 \times 10^6}\right)^{\frac{1}{2}}$$

$$= (56.73 \times 10^{+5})^{1/2}$$

$$= (5.673 \times 10^6)^{1/2}$$

$$v_e = 2.38 \times 10^3$$

$$v_e = 2.38 \times 10^3 \text{ m-sec}^{-1}$$

which is almost 1/5th of escape velocity of earth (11.2 km sec⁻¹)

+2

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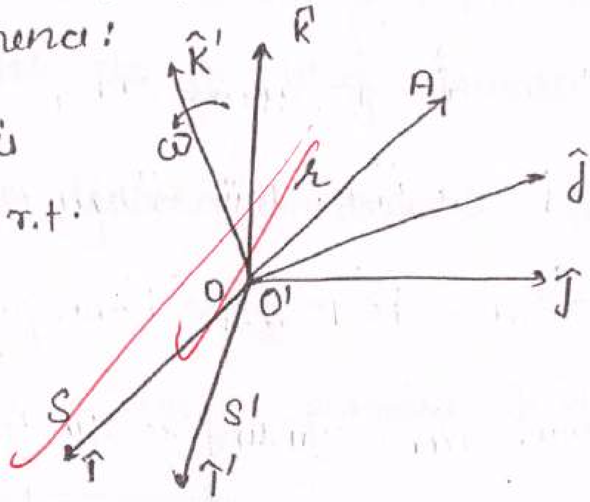
Question 3

Ans 3 (a) Equation of motion of particle in rotating frame of reference:

Let the particle is at point A w.r.t. origin of two frames S and S'.



We know that for any rotating frame of reference $|\vec{OA}| = |\vec{O'A}| = |\vec{r}|$

also the coordinates are related in following manner:

$$\left.\frac{d(\ )}{dt}\right|_{\text{Inertial}} = \left.\frac{d(\ )}{dt}\right|_{\text{Body frame}} + \vec{\omega}\times(\ )$$

for velocity of particle:

$$\left.\frac{dv}{dt}\right|_{S} = \left.\frac{dv}{dt}\right|_{S'} + \vec{\omega}\times\vec{v}$$

$$= \frac{d}{dt}(\vec{v}' + \vec{\omega}\times\vec{r}) + \vec{\omega}\times(\vec{\omega}\times\vec{r} + v')$$

$$= \frac{d\vec{v}'}{dt} + \left.\frac{d\vec{\omega}}{dt}\times\vec{r}\right|_{S'} + \left.\vec{\omega}\times\frac{d\vec{r}}{dt}\right|_{S'} + \vec{\omega}\times(\vec{\omega}\times\vec{r} + \vec{\omega}\times v'\vec{r})$$

$$\left.\frac{d\vec{\omega}}{dt}\right|_S = \left.\frac{d\vec{\omega}'}{dt}\right|_{S'} + 2(\vec{\omega}\times\vec{v}') + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$\because \frac{d\vec{\omega}}{dt} = 0$ (constant ω)

so $\vec{a}_S = \vec{a}_{S'} + 2(\vec{\omega}\times\vec{v}') + \vec{\omega}\times(\vec{\omega}\times\vec{r})$

or $m\vec{a}_S = m\vec{a}_S' + 2(\vec{\omega}\times\vec{v}')m + m\vec{\omega}\times(\vec{\omega}\times\vec{r})$

or $\boxed{\vec{F}_{real} = \vec{F} + \vec{F}_{cor} + \vec{F}_{centrifugal}}$

Here, $\vec{F}_{cor}$ = coriolis force due to rotation

$$\boxed{\vec{F}_{cor} = -2(\vec{\omega}\times\vec{V}')m}$$

$\vec{F}_{cent} = -(\vec{\omega}\times(\vec{\omega}\times\vec{r}))m$ is centrifugal force.

⇒ Both the forces are fictitous forces due to non-inertial frame (rotational)

⊙ Particle dropped from height 'h'

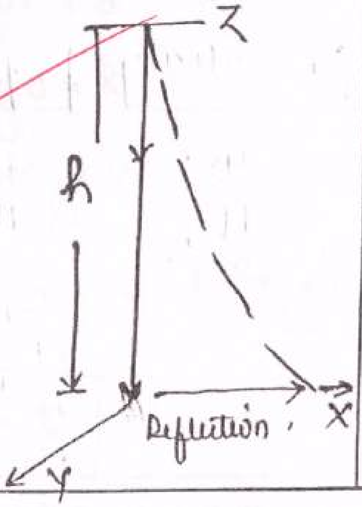
Equation of motion :

$\ddot{z} = -g$

or $\dot{z} = ~~constant~~ -gt$

$\frac{dz}{dt} = -gt$

$dz = -g t\,dt \Rightarrow z = \frac{-gt^2}{2} + c$



$z = -\frac{1}{2}gt^2 + c$, At $t=0$, $z = h$

$$\boxed{z = h - \frac{1}{2}gt^2}$$

$F_c = -2m(\vec{\omega} \times \vec{v}')$
$= -2m[\vec{\omega} \times (-v\hat{k})]$

$\because \vec{\omega} = \omega\cos\lambda\,\hat{j} + \omega\sin\lambda\,\hat{k}$

so $\vec{\omega} \times \vec{v}' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \omega\cos\lambda & \omega\sin\lambda \\ 0 & 0 & v \end{vmatrix}$

$= \hat{i}[\omega v\cos\lambda]$

So $\boxed{F_c = 2m\omega v\cos\lambda\,\hat{i}}$

Deflection: will be in $+x$ direction or towards east.

$m\frac{d^2x}{dt^2} = 2m\omega v\cos\lambda \Rightarrow \ddot{x} = 2\omega v\cos\lambda$

$\ddot{x} = 2\omega gt\cos\lambda$ $(\because v = gt)$

$\dot{x} = \omega g t^2\cos\lambda$

$\dot{x} = \omega g t^2\cos\lambda$

$x = \frac{1}{3}\omega g t^3\cos\lambda$

$\because \frac{1}{2}gt^2 = h \Rightarrow t = \sqrt{\frac{2h}{g}} \Rightarrow x = \frac{1}{3}\omega g\left(\frac{2h}{g}\right)^{3/2}\cos\lambda$

$$\boxed{x = \frac{1}{3}\omega g\left(\frac{2h}{g}\right)^{3/2}\cos\lambda}$$

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for 1Km Height and $\lambda = 0°$,

$$x = \frac{\omega g}{3}\left(\frac{2h}{g}\right)^{3/2}$$

$$= \frac{2\pi}{365 \times 24 \times 60 \times 60} \times \frac{1}{3} \times \frac{(2 \times 1 \times 10^3)^{3/2}}{\sqrt{9.8}} \text{ m}$$

$$= \frac{17.76 \times (10^3)^{3/2}}{3\sqrt{9.8}} \times 10^{-6}$$

$$= 2.16 \times 10^0$$

$$x = 2.16 \text{ metre}$$

प्रश्न संख्या (Question No.)

DIAS

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Ans 3(b) Given: $r = ae^{\theta\cos\alpha}$

$$u = \frac{1}{r} = \frac{1}{a}e^{-\theta\cos\alpha}$$

The equation of force: $\frac{d^2u}{d\theta^2} + u = -\frac{Fm}{J^2u^2}$

Now $\frac{du}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{a}e^{-\theta\cos\alpha}\right) = \frac{-\cos\alpha}{a}e^{-\theta\cos\alpha}$

$$\frac{d^2u}{d\theta^2} = \frac{\cos^2\alpha}{a}e^{-\theta\cos\alpha}$$

or $\frac{d^2u}{d\theta^2}+u = e^{-\theta\cos\alpha}\frac{\cos^2\alpha}{a} + \frac{1}{a}e^{-\theta\cos\alpha} = \frac{-Fm}{J^2u^2}$

$$\frac{e^{-\theta\cos\alpha}}{a}\left[1+\cos^2\alpha\right] = -\frac{Fm}{J^2u^2}$$

$$\Rightarrow \quad u(1+\cos^2\alpha) = -\frac{Fm}{J^2u^2}$$

$$\Rightarrow \quad F(u) = (1+\cos^2\alpha)\frac{J^2u^3}{m}$$

$$\boxed{F(r) = (1+\cos^2\alpha)\frac{J^2}{mr^3}}$$

Hence $\vec{F}(r) = (1+\cos^2\alpha)\frac{J^2}{mr^3}\hat{r}$

or $\boxed{\vec{F}(r) = (\text{constant})\frac{\hat{r}}{r^3}}$

which is required Force law.

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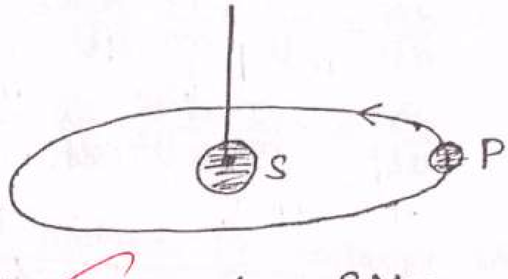
Ans 3(c) A planet with mass M is moving under gravitational attraction of sun,



The kinetic energy of planet :

$K = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$

Potential energy $U = -\frac{K}{r}$ where $k = -GMm$

Lagrangian of Planet : $L = K - U$

$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}$

or $\frac{\partial L}{\partial r} = mr\dot{\theta}^2$ and $\frac{\partial L}{\partial \dot{r}} = m\dot{r} \Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$

and $\frac{\partial L}{\partial \theta} = 0$ and $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}(mr^2\dot{\theta})$

or EQUATION OF MOTIONS.

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \Rightarrow m\ddot{r} - mr\dot{\theta}^2 = 0 \longrightarrow (1)$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{d}{dt}(mr^2\dot{\theta}) = 0$

or $\frac{dJ}{dt} = 0$ or $J = \text{constant}$ $\longrightarrow (2)$

Solving eqn (1) :

$m\ddot{r} - mr\dot{\theta}^2 = 0$ or $m\frac{d^2r}{dt^2} - mr\left(\frac{d\theta}{dt}\right)^2 = 0$.

let $u = \frac{1}{r}$ or $r = \frac{1}{u} \Rightarrow \frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt}$

or $J = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{Ju^2}{m}$.

so $\frac{dr}{dt} = -\frac{1}{u^2} \times \frac{Ju^2}{m}\frac{du}{d\theta} = -\frac{J}{m}\frac{du}{d\theta}$

$\frac{d^2r}{dt^2} = -\frac{J}{m}\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -\frac{J^2u^2}{m^2}\frac{d^2u}{d\theta^2}$.

So equation of motion becomes:

$$\frac{d^2u}{d\theta^2} + u = -\frac{F(u)m}{J^2u^2}$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{km}{J^2}$$

Let $u = \frac{km}{J^2} + A\cos(\theta - \theta_0) \longrightarrow (3)$

Energy $E = K + U$

$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \left(-\frac{K}{r}\right)$

$= \frac{1}{2}m\left[\left(\frac{du}{d\theta}\right)^2\frac{J^2}{m^2} + \frac{J^2u^4}{m^2u^2}\right] - Ku$

$E = -Ku + \frac{m}{2}\frac{J^2}{m^2}\left(\frac{du}{d\theta}\right)^2 + \frac{J^2u^2}{m^2}$

$$E = -Ku + \frac{J^2}{2m}\left(\frac{du}{d\theta}\right)^2 + \frac{J^2u^2}{m^2} \longleftarrow (4)$$

Putting $u = -\frac{km}{J^2} + A\cos(\theta - \theta_0)$ in eqn (4)

and solving we get,

$$A^2 = \frac{2mE}{J^2} + \frac{k^2m^2}{J^4}$$

or $A^2 = \frac{2mE}{J^2} + \frac{K^2m^2}{J^4} = \frac{K^2m^2}{J^4}\left[1 + \frac{2EJ^2}{mK^2}\right]$

$A = \frac{Km}{J^2}\left[1 + \frac{2EJ^2}{mK^2}\right]^{1/2}$

Hence $u = \frac{Km}{J^2} + \frac{Km}{J^2}\left(1 + \frac{2EJ^2}{mK^2}\right)^{1/2} \cos(\theta - \theta_0)$

$u = \left(\frac{Km}{J^2}\right)\left[1 + \sqrt{1 + \frac{2EJ^2}{mK^2}}\cos(\theta-\theta_0)\right]$

or $\frac{J^2/mK}{r} = 1 + \sqrt{1 + \frac{2EJ^2}{mK^2}}\cos(\theta - \theta_0)$

$$\boxed{\frac{l}{r} = 1 + e\cos(\theta - \theta_0)}$$

where l = Semi-Latus Rectum $= \frac{b^2}{a} = \frac{J^2}{mK}$

e = eccentricity $= \sqrt{1 + \frac{2EJ^2}{mK^2}}$.

CONDITIONS

for (1) $e = 0$; or $E = -\frac{mK^2}{2J^2}$; orbit is circular or bound.

(2) $e = 1$ or $E = 0 \Rightarrow$ Parabola orbit

(3) $e < 1$ or $1 + \frac{2EJ^2}{mK^2} < 1$ or $E < 0 \Rightarrow$ Elliptical orbit.

(4) $e > 1$; $E > 0$ unbound Hyperbola.

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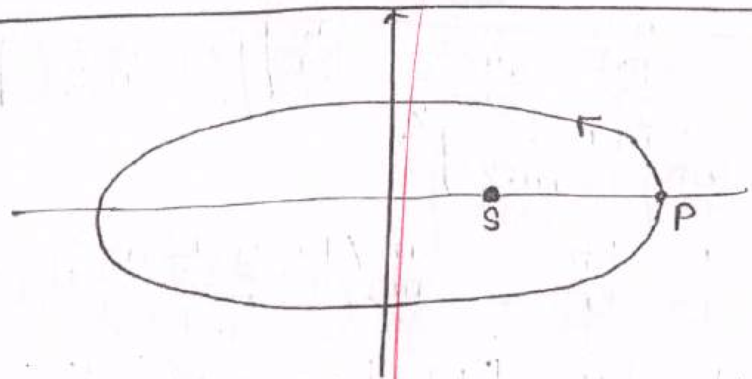


Hence for bound state ($E<0$) the orbit is elliptical or circular. Mostly the orbits are elliptical with SUN at its focus.

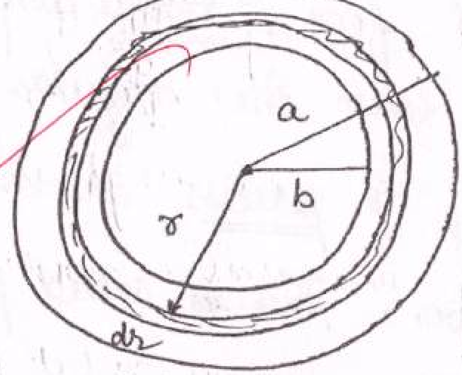
Question-4

Ans 4(a) Moment of inertia of spherical shell:

Let the density of shell be:

$\rho = \dfrac{M}{\frac{4}{3}\pi(a^3-b^3)}$

$\boxed{\rho = \dfrac{3M}{4\pi(a^3-b^3)}}$



Spherical shell

considering a shell within the range $b \le r \le a$ of thickness dr and radius 'r'.

the mass of shell $dm = \rho \times 4\pi r^2 dr$.

Moment of inertia $dI = \frac{2}{3} dm \cdot r^2$
$= \frac{2}{3}\rho 4\pi r^4 dr$.

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(∵ for a shell of negligible thickness $I = \frac{2}{3}MR^2$)

$\Rightarrow \quad I = \int dI = 4\pi\rho \cdot \frac{2}{3}\int_b^a r^4 dr$

$= \frac{8\pi}{3} \times \frac{3M}{4\pi(a^3-b^3)} \cdot \frac{r^5}{5}\Big|_b^a$

$\boxed{I = \frac{2M}{5}\left(\frac{a^5-b^5}{a^3-b^3}\right)}$

So moment of inertia of shell $= \frac{2}{5}M\frac{(a^5-b^5)}{(a^3-b^3)}$

Ans

Ans 4(b) when a rotating body is forced under an external force, torque comes into being, which forces the angular momentum of body to precess around the axis of rotation. This is called precession.

Euler's equation of motions of a rigid body rotating around its axis:

$$I_1\dot{\omega}_1 - (I_2-I_3)\omega_2\omega_3 = N_1 = 0$$
$$I_2\dot{\omega}_2 - (I_3-I_1)\omega_3\omega_1 = N_2 = 0$$
$$I_3\dot{\omega}_3 - (I_1-I_2)\omega_1\omega_2 = N_3 = 0$$

$\because \vec{N} = \vec{r}\times\vec{F} = 0$ and hence N_1, N_2, N_3 are all zero,

For a symmetric body $I_1 = I_2$ and I_3:

$$I_1\dot{\omega}_1 - (I_1-I_3)\omega_2\omega_3 = 0,$$
$$I_2\dot{\omega}_2 - (I_3-I_1)\omega_3\omega_1 = 0,$$
$$I_3\dot{\omega}_3 = 0 \quad \text{or} \quad \boxed{\omega_3 = \text{const}}$$

let $\frac{(I_1 - I_3)}{I_1}\omega_3 = \Omega$

so; equations become:

$\dot{\omega}_1 - \Omega\omega_2 = 0 \Rightarrow \ddot{\omega}_1 - \Omega\dot{\omega}_2 = 0 \longrightarrow (1)$

$\dot{\omega}_2 + \Omega\omega_1 = 0 \Rightarrow \ddot{\omega}_2 + \Omega\dot{\omega}_1 = 0 \longrightarrow (2)$

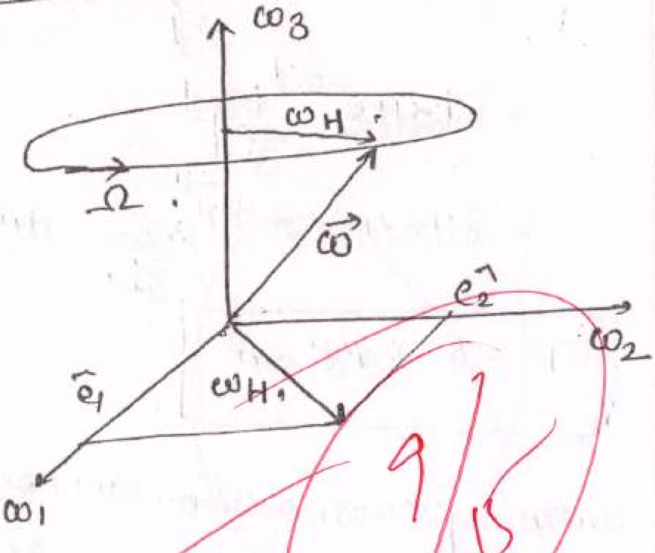
from (1) $\ddot{\omega}_1 + \Omega^2\omega_1 = 0 \Rightarrow$ $\boxed{\omega_1 = A\sin\Omega t}$

$\ddot{\omega}_2 + \Omega^2\omega_2 = 0 \Rightarrow$ $\boxed{\omega_2 = A\cos\Omega t.}$

or $\vec{\omega} = \omega_1\hat{e}_1 + \omega_2\hat{e}_2$

$\boxed{\vec{\omega}_H = A\sin\Omega t\,\hat{e}_1 + A\cos\Omega t\,\hat{e}_2}$

Hence,
the body precesses around the third axis ω_3 with precessional frequency,



$\boxed{\Omega = \frac{I_1 - I_3}{I_1}\omega_3}$ with time period $\boxed{T = \frac{2\pi}{\Omega}}$

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Ans 4(c) Given Scattering cross-section $\sigma(\theta) = 0.1 e^{-0.1} \cos^2\theta \, m^2$

Total scattering cross-section $\sigma_T = \int \sigma(\theta) d\Omega$

$$\sigma_T = \int_0^{\pi} 0.1 e^{-0.1} \cos^2\theta \cdot 2\pi \sin\theta d\theta$$

$$= 0.1 e^{-0.1} \cdot 2\pi \int_0^{\pi} \cos^2\theta \sin\theta d\theta$$

let $\cos\theta = t \Rightarrow \sin\theta d\theta = -dt$

so $\sigma_T = 2\pi \times 0.1 e^{-0.1} \int_{-1}^{1} t^2 dt$

$$= 0.1 \times 2\pi e^{-0.1} \left. \frac{t^3}{3} \right|_{-1}^{1}$$

$$= 0.1 \times 2\pi \times e^{-0.1} \times \frac{2}{3} \quad m^2$$

$$\boxed{\sigma_T = 0.378 \ m^2}$$

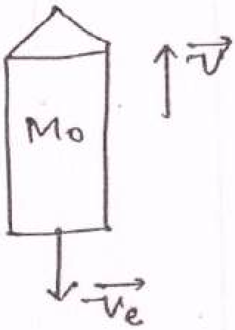
Scattering cross section is no. of particles scattered per unit time per unit incident flux.

Ans 4(c) Rocket is a variable-mass system working on Newton's third law of motion.

Given: mass of rocket $m_0 = 10^3$ kg

Rate of expulsion $\alpha = 2$ kg-sec^{-1}

(i) Minimum velocity for rocket to rise :—



Rocket equation ;

$$M \frac{dv}{dt} = -gm - v_e \frac{dm}{dt}$$

For Rocket to just begin to rise:

$\frac{dv}{dt} = 0$; $\quad gm = -v_e \frac{dm}{dt}$

$$\Rightarrow (v_e)_m = \frac{-gm}{dm/dt}$$

$$= \frac{1000 \times 9.8}{2}$$

$$\boxed{(v_e)_m = 4.9 \text{ km/sec}}$$

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(ii) Assuming minimum exhaust velocity.

after t=10 sec: Mass $m = m_0 - \alpha t$

$= 1000 - 2 \times 10$

$m = 980$ kg

so $v(t) = v_0 - gt + \left(\ln \frac{m_0}{m}\right) v_e$

here $v_0 = 0$ at $t = 0$;

$$v(t=10) = -9.8 \times 10 + 4900 \ln \frac{1000}{980}$$

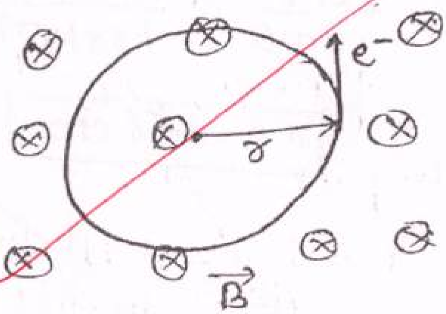
$$= -98 + 98.99$$

$$\boxed{v = 0.99 \text{ m/sec}}$$

ANSWER

Question 5

Ans 5(a) The cyclotron is a device to accelerate charged particles using external magnetic field.



Given: Potential difference

$V = 10$ Mega Volt

Energy of e^- $E = qV$

$$E = 10 \text{ MeV}$$

Applied $\vec{B} = -2$ Tesla $\hat{k}$ (say the direction is in $-z$ axis)

$|\vec{B}| = 2T$

As the e^- gets accelerated in a circular path subjected to centripetal force:

$$\frac{mv^2}{r} = qvB.$$

$$\text{or } r = \frac{mv^2}{qvB} = \frac{mv}{qB}.$$

$$r = \frac{mv}{qB} \longrightarrow (1)$$

The e^- energy $E = 10 \text{ MeV} >> m_0c^2$ (0.511 MeV)

So $E^2 = p^2c^2 + m_0^2c^4 \Rightarrow p^2c^2 = E^2 - m_0^2c^4$

$= 100 - 0.26 \text{ MeV}^2$

$p^2c^2 = 100\ MeV^2$

$pc = 10\ MeV \Rightarrow p = \frac{10 \times 1.6 \times 10^{-13}}{3 \times 10^8}$

$\boxed{p = 5.3 \times 10^{-21}}$ kg-m/sec

so $r = \frac{p}{qB} = \frac{5.3 \times 10^{-21}}{1.6 \times 10^{-19} \times 2} = 1.66 \times 10^{-2}$

$\boxed{r = 1.66\ cm}$

7/10

Ans 5(b) Young's modulus is the ratio of longitudinal stress(S) to longitudinal strain (s)

$\boxed{Y = \frac{S}{s}}$

The modulus of rigidity (η) is ratio of tangential stress to shear strain.

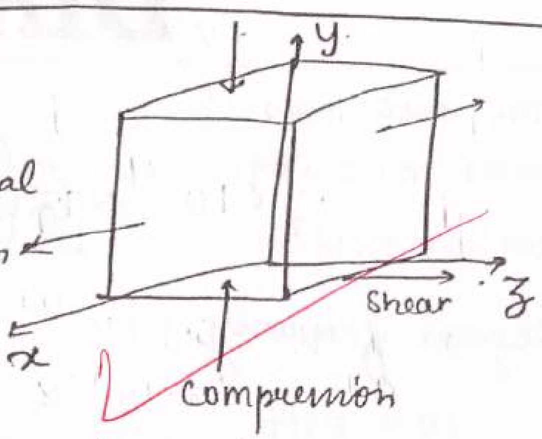
$\boxed{\eta = \frac{P}{\theta}}$

Lateral strain to ~~latter~~ longitudinal strain ratio is called Poison Ratio (σ).

or $\sigma = \frac{S_{lat}}{S_{Long}} \Rightarrow S_{lat} = \sigma \frac{\Delta L}{L} = \frac{\sigma P}{Y}$

so $\boxed{\Delta d = \frac{\sigma P}{Y} L}$ —— (1)

Applying expansional force along x-axis (in both directions) & compressional force in Y-direction of a cube.



$$\text{Net Shear} = \underbrace{\left|\frac{L_x - L}{L}\right|}_{\text{expansion}} + \underbrace{\left|\frac{L_y - L}{L}\right|}_{\text{compression}}$$

Now, $\frac{L_x - L}{L} = \frac{P}{Y} + \frac{\sigma P}{Y} \Rightarrow \frac{L_x - L}{L} = \frac{P}{Y}(1+\sigma)$

$$\frac{L_Y - L}{L} = \frac{P}{Y} + \frac{\sigma P}{Y} = \frac{P}{Y}(1+\sigma)$$

$$\text{Net} = \frac{2P}{Y}(1+\sigma).$$

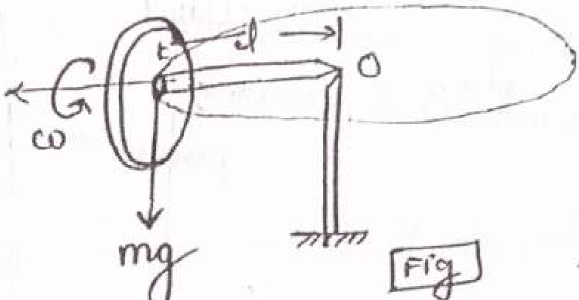
$$\theta = \frac{2P}{Y}(1+\sigma) \Rightarrow \frac{P}{\theta} = \frac{Y}{2(1+\sigma)} \text{ or } \eta = \frac{Y}{2(1+\sigma)}$$

$$\Rightarrow \boxed{Y = 2\eta(1+\sigma)}$$ hence proved.

6½/7½

Ans 5(c) The disc has mass $m = 0.5\,kg$

radius $= 5\times10^{-2}\,m$



Fig

Angular frequency:

$\omega = 2\pi f$

$= 2\pi \times 50 = 100\pi$ Radian.

$l = 3\times10^{-2}\,m.$

Now $\because$ $\Omega\Delta t = \dfrac{\Delta J}{J\sin\theta}$ $\Rightarrow \Omega = \dfrac{\Delta J/\Delta t}{J\sin\theta}$

$\Omega = \dfrac{\text{Torque}}{J\sin\theta} = \dfrac{mgl\sin\theta}{J\sin\theta}$

Precessional frequency $\Omega = \dfrac{mgl}{J} = \dfrac{mgl}{I\omega} = \dfrac{mgl}{\frac{1}{2}MR^2\omega}$

or $\boxed{\Omega = \dfrac{2gl}{R^2\omega}}$

7/10

Putting values: $\Omega = \dfrac{2\times9.8\times3\times10^{-2}}{(5\times10^{-2})^2\times100\times\pi}$

$= 7.5\times10^{-3}\times10^{2}$

$= 7.5\times10^{-1}$

$\Rightarrow T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{0.75} = 8.37$ sec.

Hence time period of Horizontal precessional motion $\boxed{T = 8.37 \text{ Sec}}$

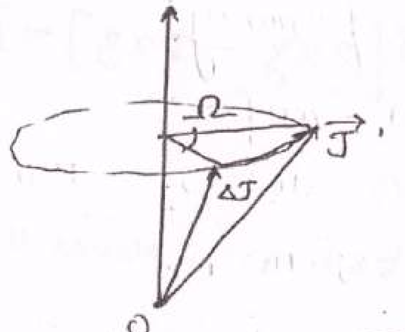


Fig. Precession of $\vec{J}$

Ans 5 (d) (i) $\vec{F}$ is said to be conservative when the work done on displacing particle solely depends upon the end points of path and not the path itself.

(or) $\oint \vec{F} \cdot \vec{dr} = 0$ or $\boxed{\vec{\nabla} \times \vec{F} = 0}$

Given $\vec{F} = (2xy + yz^2)\hat{i} + (x^2 + xz^2)\hat{j} + 2xyz\hat{k}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+yz^2 & x^2+xz^2 & 2xyz \end{vmatrix}$$

$$= \hat{i}[2xz - 2xz] - \hat{j}[2yz - 2yz] + \hat{k}[2x + z^2 - 2x - z^2]$$

$$= 0$$

So, $\vec{\nabla}\times\vec{F} = 0$ Hence the force is conservative.

(ii) $\vec{F} = yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix}$$

$$= \hat{i}[x - x] - \hat{j}[y - y] + \hat{k}[z - z]$$

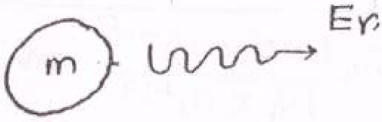
$$= 0$$

$\boxed{\vec{\nabla}\times\vec{F} = 0}$ The force is conservative.

out of both forces given **No force is** ~~no~~ NON-CONSERVATIVE.

6½/10

Ans 5(e)



γ-emission of any mass creates a recoil due to conservation of angular momentum.

conservation of energy :-

$$E_m = E_R + E_\gamma$$

$$mc^2 = E_R + E_\gamma$$

$$\boxed{E_R = mc^2 - E_\gamma}$$

Now considering mass at rest initially total rest mass energy $E = mc^2$.

Conservation of momentum :

$$P_m = P_\gamma$$

$$P_m = \frac{E_\gamma}{c} \quad (\because E_\gamma = Pc \text{ as Rest mass } \gamma = 0)$$

$$= \frac{10\ MeV}{3\times10^8}$$

$$= \frac{10\times1.6\times10^{-19}\times10^6}{3\times10^8}$$

$$\boxed{P_m = 5.33\times10^{-21}\ kg\,m\text{-}sec^{-1}}$$

Hence Recoil energy $E_R = \frac{P^2}{2m} = \frac{(5.33\times10^{-21})^2}{2\times10^{-23}}$

$$E_R = 14.2\times10^{-19}\ J$$

or $E_R = \frac{1.42 \times 10^{-18}}{1.6 \times 10^{-19}}$ eV

$E_R = 8.875$ eV